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Quantisation of θ-expanded non-commutative QED

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Abstract. We analyse two new versions of θ-expanded non-commutative quantum electrodynamics up to first order in θ and first loop order. In the first version we expand the bosonic sector using the Seiberg–Witten map, leaving the fermions unexpanded. In the second version we leave both bosons and fermions unexpanded. The analysis shows that the Seiberg–Witten map is a field redefinition at first order in θ. However, at higher order in θ the Seiberg–Witten map cannot be regarded as a field redefinition. We find that the initial action of any θ-expanded massless non-commutative QED must include one extra term proportional to θ which we identify by loop calculations.

1 Introduction

Quantum field theory on non-commutative structures has received increasing attention during the last years [1–4]. In almost all articles on the subject a non-commutative structure

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \tag{1.1}$$

characterised by a constant non-commutativity parameter θ has been considered, mainly due to the possibility of explicit calculations. Some investigations of field theories involving a non-constant θ have been performed [5,6]. In any case, non-commutative configurations of the form (1.1) are to be seen in the spirit of deformation quantisation [7], which form a subspace of all possible non-commutative settings [8]. The algebra (1.1) with constant θ serves as the simplest possible setting for a non-commutative quantum field theory.

Because the relation (1.1) implies a non-locality of the underlying space-time, the question of renormalisability and, as a consequence hereof, the method of quantisation, is of central interest.

The method mostly applied for quantisation takes the full, non-expanded, non-local action as the source of Feynman rules. This leads to damping phases which renders the non-planar sector UV-finite. However, the entailed UV–IR mixing leads to new infrared divergences [9] which spoil renormalisability beyond first loop order. A thorough analysis of the problem was given in [10,11]. In the special case of non-commutative ϕ^4-theory methods of finite temperature quantum field theory have been used to resum the perturbation series leading to renormalisability [12]. This method might be useful for non-commutative gauge theories as well. So far, however, the problem of preserving the gauge symmetry has not been solved. Ideas involving field redefinitions to overcome the IR-problems were presented in [13]. It is however unclear whether this idea will prove fruitful because it transfers the problems to higher n-point functions rather than removing them from the theory. The only solution to the problem of quantising non-commutative gauge theory known today is the introduction of supersymmetry [14,15], because the divergences present in certain supersymmetric models are "soft" enough to render the UV–IR mixing integrable [16].

An alternative method of quantisation was proposed in [17]: The non-commutativity structure θ apparently limits the choice of gauge group to that of a matrix representation of a $U(N)$ gauge group. The choice of a more general group automatically entails enveloping algebra valued fields rendering the model seemingly meaningless. This problem is solved by expanding the model in θ via the Seiberg–Witten map [1,18,19], which expresses the non-commutative gauge field in terms of a commutative gauge field. The price paid is however very high: θ-expanded theories are truly power-counting non-renormalisable and involve infinitely many vertices with an arbitrary number of legs. In [20] the purely bosonic case was analysed for an Abelian group and in [21] it was shown that the photon self-energy is renormalisable to all orders. In [22], however, θ-expanded non-commutative QED was proven non-renormalisable, putting this line of quantisation to an apparent halt.

A central problem related to non-commutative field theories which becomes urgently important in the second method of quantisation is the choice of the action: In order

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to quantise a power-counting non-renormalisable model one needs very strong symmetries. Symmetries are to be found in the classical action, but the above scenario does not give any criteria which dictate the form of the action, apart from Lorentz and gauge symmetries and the demand that the limit $\theta \to 0$ yields the commutative model. These bonds leaves a huge space of possible actions. If the naive – initial – choice of the commutative action equipped with appropriate star products should prove renormalisable the non-commutativity must somehow yield symmetries by itself. On the other hand, one could speculate whether the quantisation procedure will by itself cast light on this question by forcing extra terms to the initial naive choice and thereby lead us to a suitable action. In the light of recent phenomenological works considering the θ-expanded standard model via the Seiberg–Witten map [23] we find it important to single out the initial action by the behaviour of the Seiberg–Witten map on quantum level.

In this paper we analyse two variations of the second method of quantisation (θ-expansion) up to first order in θ and first loop order. These two variations are closely related to the θ-expanded non-commutative QED studied in [22] where both the bosonic and fermionic sectors were expanded via the Seiberg–Witten map. Here we first consider θ-expanded non-commutative QED in the special case where the Seiberg–Witten map is only applied to the bosons. The reason for this is speculative: Whereas there are strong mathematical reasons for expanding the bosons via the Seiberg–Witten map [18] there do not seem to be urgent reasons for applying the Seiberg–Witten differential equation to the fermions [24]. Secondly, we consider the case of θ-expanded non-commutative QED without use of the Seiberg–Witten map whatsoever. In connection to the above it is natural to investigate this case in order to fully understand the role of the Seiberg–Witten map.

What we find is partly encouraging: Up to unphysical field redefinitions both studied settings coincide with the results of [22]. This means that the Seiberg–Witten map is nothing but a field redefinition at first order in θ. However, we find substantial evidence that this will not be the case at higher orders in θ, thus leaving a small window of hope open for θ-expanded models. It means, though, that the initial action of *any* θ-expanded model involving fermions must be extended with at least one extra term which we identify. This extra term suffices – in the massless case – for one-loop renormalisability at first order in θ.

In the massive case two extra instabilities occur in the fermionic sector, which cannot be removed in an obvious way. It thus appears that we are cumulating evidence that *if* one insists on considering θ-expanded non-commutative Yang–Mills theory, then the fermionic mass must be introduced via a Higgs mechanism.

Let us mention that our initial motivation for studying θ-expanded field theories without using the Seiberg–Witten map was first of all the wish to obtain θ-graded symmetries which could fix the θ-structure of the action on the quantum level. It turned out, however, that this does not work because we loose at the same time the initial "flat" gauge symmetry. One could speculate if other symmetries present in the original θ-non-expanded action (we think e.g. of symmetries of the spectral action [25]; see also the discussion in [22]) could provide useful θ-graded symmetries of the expanded action. An interesting example is supersymmetry which indeed yields such a θ-graded symmetry [26].

This paper is organised as follows: In Sect. 2 we introduce non-commutative Yang–Mills theory. In Sect. 3 we expand the action using the Seiberg–Witten map in the bosonic sector only. The quantisation is studied at first loop order. In Sect. 4 we repeat the analysis for the action expanded in θ without the Seiberg–Witten map. In Sect. 5 we analyse general changes of variables in the path-integral and, finally, in Sect. 6 we give a conclusion.

2 Non-commutative Yang–Mills theory

On the space of rapidly decreasing functions on $\mathbb{R}^4$ one introduces a deformed product

$$
\begin{aligned}
&(f \star g)(x) \qquad (2.1)\\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}(k_\mu+p_\mu)x^\mu} \mathrm{e}^{-(\mathrm{i}/2)\theta^{\mu\nu}k_\mu p_\nu} \tilde{f}(k)\, \tilde{g}(p),
\end{aligned}
$$

where f, g are rapidly decreasing functions on the manifold and $\tilde{f}, \tilde{g}$ their Fourier transforms[1]. The $\star$-product (2.1) is associative and non-commutative and yields for the coordinates x^μ belonging to the multiplier algebra the commutator

$$
[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}. \qquad (2.2)
$$

We consider the action of non-commutative Yang-Mills (NCYM) theory, including fermions, given by

$$
\hat{\Sigma}_{\mathrm{cl}} = \int \mathrm{d}^4x \left(-\frac{1}{4g^2} \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} + \widehat{\overline{\psi}} \star \mathrm{i}\gamma^\mu \hat{D}_\mu \hat{\psi} - m\widehat{\overline{\psi}} \star \hat{\psi} \right), \qquad (2.3)
$$

with

$$
\begin{aligned}
\hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}[\hat{A}_\mu, \hat{A}_\nu]_\star,\\
\hat{D}_\mu \hat{\psi} &= \partial_\mu \hat{\psi} - \mathrm{i}\hat{A}_\mu \star \hat{\psi}.
\end{aligned}
$$

Notice that this is a *non-local* field theory. The action (2.3) is invariant with respect to an infinitesimal gauge transformation

$$
\begin{aligned}
\delta_{\hat{\Lambda}} \hat{A}_\mu &= \hat{D}^{\mathrm{adj}}_\mu \hat{\Lambda} \equiv \partial_\mu \hat{\Lambda} - \mathrm{i}[\hat{A}_\mu, \hat{\Lambda}]_\star,\\
\delta_{\hat{\Lambda}} \hat{\psi} &= \mathrm{i}\hat{\Lambda} \star \hat{\psi},\\
\delta_{\hat{\Lambda}} \widehat{\overline{\psi}} &= -\mathrm{i}\widehat{\overline{\psi}} \star \hat{\Lambda}. \qquad (2.4)
\end{aligned}
$$

Usually this gauge symmetry is fixed via a gauge-fixing term, introducing ghost $\hat{c}$, anti-ghost $\widehat{\overline{c}}$ and multiplier field $\widehat{B}$,

$$
\hat{\Sigma}_{\mathrm{gf}} = \int \mathrm{d}^4x\hat{s} \left(\widehat{\overline{c}} \star \partial^\mu \hat{A}_\mu + \frac{\alpha}{2} \widehat{\overline{c}} \star \hat{B} \right), \qquad (2.5)
$$

[1] Our Fourier conventions are $f(x) = \int (\mathrm{d}^4k)/((2\pi)^4)\mathrm{e}^{-\mathrm{i}k\cdot x} \tilde{f}(k)$ and $\tilde{f}(k) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}k\cdot x} f(x)$

where we define the non-commutative BRST transformations by

$$\begin{aligned}\hat{s}\hat{A}_\mu &= \hat{D}^{\mathrm{adj}}_\mu \hat{c}, \quad \hat{s}\hat{c} = \mathrm{i}\hat{c}\star\hat{c},\\ \hat{s}\hat{\psi} &= \mathrm{i}\hat{c}\star\hat{\psi}, \quad \hat{s}\hat{\bar{\psi}} = -\mathrm{i}\hat{\bar{\psi}}\star\hat{c},\\ \hat{s}\hat{\bar{c}} &= \hat{B}, \quad \hat{s}\hat{B} = 0.\end{aligned} \tag{2.6}$$

Finally, we couple the non-linear BRST transformations to external fields $(\hat{\bar{\rho}}, \hat{\rho}, \hat{\sigma}^\mu, \hat{\kappa})$ by introducing an extra term into the action,

$$\hat{\Sigma}_{\mathrm{ext}} = \int \mathrm{d}^4x(\hat{\bar{\rho}}(\hat{s}\hat{\psi}) + (\hat{s}\hat{\bar{\psi}})\hat{\rho} + \hat{\sigma}^\mu(\hat{s}\hat{A}_\mu) + \hat{\kappa}(\hat{s}\hat{c})), \tag{2.7}$$

and define

$$\hat{s}\hat{\rho} = 0, \quad \hat{s}\hat{\bar{\rho}} = 0, \quad \hat{s}\hat{\sigma}^\mu = 0, \quad \hat{s}\hat{\kappa} = 0. \tag{2.8}$$

The full tree level generating functional for 1PI graphs reads

$$\hat{\Gamma}^{(0)} = \left(\hat{\Sigma}_{\mathrm{cl}} + \hat{\Sigma}_{\mathrm{ext}} + \hat{\Sigma}_{\mathrm{gf}}\right). \tag{2.9}$$

The nilpotency of $\hat{s}$ allows us to write down the Slavnov–Taylor identity expressing the BRST symmetry:

$$\mathcal{S}\left(\hat{\Gamma}^{(0)}\right) = 0, \tag{2.10}$$

$$\mathcal{S}(\Gamma) = \int \mathrm{d}^4x \left(\frac{\delta\Gamma}{\delta\hat{\sigma}^\mu}\frac{\delta\Gamma}{\delta\hat{A}_\mu} + \frac{\delta\Gamma}{\delta\hat{\kappa}}\frac{\delta\Gamma}{\delta\hat{c}} + \frac{\delta\Gamma}{\delta\hat{\rho}}\frac{\delta\Gamma}{\delta\hat{\bar{\psi}}} + \frac{\delta\Gamma}{\delta\hat{\psi}}\frac{\delta\Gamma}{\delta\hat{\bar{\rho}}} + \hat{B}\frac{\delta\Gamma}{\delta\hat{\bar{c}}}\right). \tag{2.11}$$

3 Expanding the action (Case I)

In this section we expand the action of NCYM theory in θ using the Seiberg–Witten differential equation in the bosonic sector. In contrast to the analysis performed in [22] on θ-expanded QED [27] the fermions are not expanded. This entails a picture where the gauge symmetry is "flat" (not θ-graded) in the bosonic sector and "tilted" (θ-graded) in the fermionic sector. Performing the loop calculations we show that the Slavnov–Taylor identities are still valid on quantum level. Also, we find that the model characterised by the classical action (2.3) is instable. Remarkable enough, the bosonic sector is stable at first order in θ. These results are up to field redefinitions identical to the results found in [22].

3.1 Classical analysis

The expansion of the action (2.9) is performed according to

$$(f\star g)(x) = f(x)g(x) + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\partial_\alpha f(x)\partial_\beta g(x) + \mathcal{O}(\theta^2),$$

$$\begin{aligned}\hat{A}_\mu &= A_\mu - \frac{1}{2}\theta^{\rho\sigma}A_\rho\left(\partial_\sigma A_\mu + F_{\sigma\mu}\right) + \mathcal{O}(\theta^2),\\ \hat{\Phi} &= \Phi, \quad \forall\hat{\Phi}\in\{\hat{\psi}, \hat{\bar{\psi}}, \hat{c}, \hat{\bar{c}}, \hat{B}, \hat{\bar{\rho}}, \hat{\rho}, \hat{\sigma}^\mu, \hat{\kappa}\},\end{aligned} \tag{3.1}$$

where the gauge field is expanded according to the Seiberg–Witten differential equation [1]. This leads to the expanded action

$$\Sigma^{\{n\}}_{\theta-\mathrm{exp}} = \sum_{i=0}^{n}\Sigma^{(i)}, \tag{3.2}$$

which up to first order in θ (which we are interested in from now on) reads

$$\Sigma^{(0)}_{\mathrm{cl}} = \int \mathrm{d}^4x\left(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(\mathrm{i}\gamma^\mu D_\mu - m\right)\psi\right), \tag{3.3}$$

$$\begin{aligned}\Sigma^{(1)}_{\mathrm{cl}} = \int \mathrm{d}^4x\bigg(&\frac{1}{8g^2}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\\ &- \frac{1}{2g^2}\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu}\\ &+ \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\bar{\psi}\gamma^\mu\partial_\alpha A_\mu\partial_\beta\psi - \theta^{\alpha\beta}\bar{\psi}\gamma^\mu A_\alpha\partial_\beta A_\mu\psi\\ &+ \frac{1}{2}\theta^{\alpha\beta}\bar{\psi}\gamma^\mu A_\alpha\partial_\mu A_\beta\psi\bigg),\end{aligned} \tag{3.4}$$

$$\Sigma^{(0)}_{\mathrm{gf}} = \int \mathrm{d}^4x\left(B\partial^\mu A_\mu + \frac{\alpha}{2}BB - \bar{c}\partial^\mu\partial_\mu c\right), \tag{3.5}$$

$$\Sigma^{(\geq 1)}_{\mathrm{gf}} = 0. \tag{3.6}$$

We choose the "linear gauge-fixing" in the sense of [20] applied after the Seiberg–Witten map (3.1). The θ-expansion of (2.5) leads to the "non-linear gauge-fixing" which is different[2]. We expand the BRST transformations (2.4) according to (3.1),

$$\hat{s} = \sum_i s^{(i)},$$

which to first order in θ gives

$$\begin{aligned}s^{(0)}A_\mu &= \partial_\mu c, \quad s^{(0)}c = 0,\\ s^{(0)}\psi &= \mathrm{i}c\psi, \quad s^{(0)}\bar{\psi} = -\mathrm{i}\bar{\psi}c,\\ s^{(1)}A_\mu &= 0, \quad s^{(1)}c = 0,\\ s^{(1)}\psi &= -\frac{\mathrm{i}}{2}\theta^{\alpha\beta}A_\alpha\partial_\beta c\psi - \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha c\partial_\beta\psi,\\ s^{(1)}\bar{\psi} &= \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\bar{\psi}A_\alpha\partial_\beta c + \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha\bar{\psi}\partial_\beta c.\end{aligned} \tag{3.7}$$

Here we have used the Seiberg–Witten expansion of the non-commutative gauge parameter [1]

$$\hat{\Lambda} = \lambda - \frac{1}{2}\theta^{\alpha\beta}A_\alpha\partial_\beta\lambda + \mathcal{O}(\theta^2), \tag{3.8}$$

[2] In Case I we apply the linear gauge, which is possible because the BRST symmetry is "flat" in the bosonic sector – in perfect analogy to [20]. In Case II (see Sect. 4) we use (a variation of) the non-linear gauge, because in Case II we have a θ-graded BRST symmetry in the bosonic sector, leaving no room for a linear gauge. Since we have shown that the choice of linear/non-linear gauge leaves loop calculations invariant [20] this is justified

replacing λ by c [20], to obtain the non-commutative gauge transformation of the fermions in terms of the commutative gauge parameter. Notice that only the BRST transformations of the fermions (3.7) are θ-graded. The application of the Seiberg–Witten map in the bosonic sector "flattens" out their BRST transformations. The total θ-expanded action is invariant under non-commutative BRST transformations

$$s\sum_i \Sigma^{(i)} = 0, \tag{3.9}$$

leading to a tower of symmetries

$$s^{(0)}\Sigma^{(0)} = 0, \tag{3.10}$$
$$s^{(1)}\Sigma^{(0)} + s^{(0)}\Sigma^{(1)} = 0, \tag{3.11}$$
$$s^{(2)}\Sigma^{(0)} + s^{(1)}\Sigma^{(1)} + s^{(0)}\Sigma^{(2)} = 0, \tag{3.12}$$
$$\vdots$$

where (3.10) is simply the BRST invariance of the commutative theory.

3.2 Slavnov–Taylor identity

Loop corrections do not preserve the BRST symmetry in the form (3.9). The solution of this problem is to couple the non-linear BRST transformations to external fields,

$$\Sigma^{(n)}_{\text{ext}} = \int \mathrm{d}^4x\,(\sigma^\mu s^{(n)}A_\mu + \kappa s^{(n)}c + \bar\rho s^{(n)}\psi + s^{(n)}\bar\psi\rho)\,. \tag{3.13}$$

Defining the full tree level generating functional for 1PI graphs to nth order in θ by

$$\Gamma^{(n,0)} = \left(\Sigma^{(n)}_{\theta-\exp} + \Sigma^{(n)}_{\text{ext}} + \Sigma^{(n)}_{\text{gf}}\right), \tag{3.14}$$

the Slavnov–Taylor identity expresses the whole set of BRST invariances (3.10)–(3.12) up to nth order in θ,

$$(\mathcal{S}\Gamma)^{(n)} = 0, \tag{3.15}$$

where the Slavnov–Taylor operator is defined by (2.11) (without the hat over the fields). In momentum space we have

$$0 = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(\frac{\delta\Gamma}{\delta\sigma^\mu(k)}\frac{\delta\Gamma}{\delta A_\mu(-k)} + \frac{\delta\Gamma}{\delta\rho(k)}\frac{\delta\Gamma}{\delta\bar\psi(-k)} + \frac{\delta\Gamma}{\delta\psi(k)}\frac{\delta\Gamma}{\delta\bar\rho(-k)} + \frac{\delta\Gamma}{\delta\kappa(k)}\frac{\delta\Gamma}{\delta c(-k)} + B(k)\frac{\delta\Gamma}{\delta\bar c(k)}\right). \tag{3.16}$$

Functional derivation of (3.16) with respect to the fields $\{A_\mu, c, \psi, \bar\psi, \bar c, B\}$ in momentum space, followed by putting the fields to zero, leads to various forms of the Slavnov–Taylor identity for 1PI Green's functions

$$(2\pi)^4\delta(p_1+\ldots+p_N)\Gamma_{\Phi_1\ldots\Phi_N}(p_1,\ldots,p_N) := \frac{\delta^N\Gamma}{\delta\Phi_N(p_N)\ldots\delta\Phi_1(p_1)}\bigg|_{\Phi_i=0}. \tag{3.17}$$

These Green's functions,

$$\Gamma_{\Phi_1\ldots\Phi_N}(p_1,\ldots,p_N) = \sum_{n,\ell\geq 0}\Gamma^{(n,\ell)}_{\Phi_1\ldots\Phi_N}(p_1,\ldots,p_N),$$

carry a bidegree (n,ℓ) where n is the number of factors of θ and ℓ the number of loops. For our purpose the most important Slavnov–Taylor identities derived from (3.16) are the following:

$$\begin{aligned}0 = \sum_{\ell'=0}^{\ell}\sum_{n'=0}^{n}\Big(&\Gamma^{(n',\ell')}_{\mu;\sigma c}(q{+}r,p)\Gamma^{(n-n',\ell-\ell')\mu}_{A\bar\psi\psi}(p,q,r)\\ &+\Gamma^{(n',\ell')}_{\bar\psi c\rho}(q,p,r)\Gamma^{(n-n',\ell-\ell')}_{\bar\psi\psi}(p{+}q,r)\\ &+\Gamma^{(n-n',\ell-\ell')}_{\bar\psi\psi}(q,p{+}r)\Gamma^{(n',\ell')}_{\bar\rho c\psi}(q,p,r)\Big),\end{aligned} \tag{3.18}$$

$$\begin{aligned}0 = \sum_{\ell'=0}^{\ell}\sum_{n'=0}^{n}\Big(&\Gamma^{(n',\ell')}_{\mu;\sigma c}(q{+}r{+}s,p)\Gamma^{(n-n',\ell-\ell')\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)\\ &+\Gamma^{(n',\ell')\nu}_{\mu;A\sigma c}(q,r{+}s,p)\Gamma^{(n-n',\ell-\ell')\mu}_{A\bar\psi\psi}(p{+}q,r,s)\\ &+\Gamma^{(n',\ell')\nu}_{A\bar\psi c\rho}(q,r,p,s)\Gamma^{(n-n',\ell-\ell')}_{\bar\psi\psi}(p{+}q{+}r,s)\\ &+\Gamma^{(n-n',\ell-\ell')}_{\bar\psi\psi}(r,p{+}q{+}s)\Gamma^{(n',\ell')\nu}_{A\bar\rho c\psi}(q,r,p,s)\\ &+\Gamma^{(n',\ell')}_{\bar\psi c\rho}(r,p,q{+}s)\Gamma^{(n-n',\ell-\ell')\nu}_{A\bar\psi\psi}(q,p{+}r,s)\\ &+\Gamma^{(n-n',\ell-\ell')\nu}_{A\bar\psi\psi}(q,r,p{+}s)\Gamma^{(n',\ell')}_{\bar\rho c\psi}(q{+}r,p,s)\Big),\end{aligned} \tag{3.19}$$

$$0 = \sum_{\ell'=0}^{\ell}\sum_{n'=0}^{n}\Gamma^{(n',\ell')}_{\mu;\sigma c}(q,p)\Gamma^{(n-n',\ell-\ell')\mu\nu}_{AA}(p,q), \tag{3.20}$$

$$\begin{aligned}0 = \sum_{\ell'=0}^{\ell}\sum_{n'=0}^{n}\Big(&\Gamma^{(n',\ell')\nu}_{\mu;A\sigma c}(q,r,p)\Gamma^{(n-n',\ell-\ell')\mu\rho}_{AA}(p{+}q,r)\\ &+\Gamma^{(n',\ell')\rho}_{\mu;A\sigma c}(r,q,p)\Gamma^{(n-n',\ell-\ell')\mu\nu}_{AA}(p{+}r,q)\\ &+\Gamma^{(n',\ell')}_{\mu;\sigma c}(q{+}r,p)\Gamma^{(n-n',\ell-\ell')\mu\nu\rho}_{AAA}(p,q,r)\Big),\end{aligned} \tag{3.21}$$

$$\begin{aligned}0 = \sum_{\ell'=0}^{\ell}\sum_{n'=0}^{n}\Big(&\Gamma^{(n',\ell')}_{\mu;\sigma c}(q{+}r{+}s,p)\Gamma^{(n-n',\ell-\ell')\mu}_{A\bar\psi c\rho}(p,r,q,s)\\ &-\Gamma^{(n',\ell')}_{\mu;\sigma c}(p{+}r{+}s,q)\Gamma^{(n-n',\ell-\ell')\mu}_{A\bar\psi c\rho}(q,r,p,s)\\ &+\Gamma^{(n',\ell')}_{\bar\psi c\rho}(r,p,q{+}s)\Gamma^{(n-n',\ell-\ell')}_{\bar\psi c\rho}(p{+}r,q,s)\\ &-\Gamma^{(n',\ell')}_{\bar\psi c\rho}(r,q,p{+}s)\Gamma^{(n-n',\ell-\ell')}_{\bar\psi c\rho}(q{+}r,p,s)\\ &+\Gamma^{(n',\ell')}_{\kappa cc}(r{+}s,p,q)\Gamma^{(n-n',\ell-\ell')}_{\bar\psi c\rho}(r,p{+}q,s)\Big),\end{aligned} \tag{3.22}$$

$$0 = \sum_{\ell'=0}^{\ell}\sum_{n'=0}^{n}\Big(\Gamma^{(n',\ell')}_{\mu;\sigma c}(q{+}r{+}s,p)\Gamma^{(n-n',\ell-\ell')\mu}_{A\bar\rho c\psi}(p,r,q,s)$$

$$
\begin{aligned}
&- \Gamma^{(n',\ell')}_{\mu;\sigma c}(p+r+s,q)\Gamma^{(n-n',\ell-\ell')\mu}_{A\bar\rho c\psi}(q,r,p,s)\\
&- \Gamma^{(n',\ell')}_{\bar\rho c\psi}(r,p,q+s)\Gamma^{(n-n',\ell-\ell')}_{\bar\rho c\psi}(p+r,q,s)\\
&+ \Gamma^{(n',\ell')}_{\bar\rho c\psi}(r,q,p+s)\Gamma^{(n-n',\ell-\ell')}_{\bar\rho c\psi}(q+r,p,s)\\
&+ \Gamma^{(n',\ell')}_{\kappa cc}(r+s,p,q)\Gamma^{(n-n',\ell-\ell')}_{\bar\rho c\psi}(r,p+q,s)\Big). \qquad (3.23)
\end{aligned}
$$

For $n=0$ we recover ordinary QED, where additionally $\ell'=0$ because there are no loops involving external fields.

The above Slavnov–Taylor identities can be verified on a formal level of divergent integrals and hold for renormalised Green's functions when using an invariant regularisation scheme. However, in contrast to the commutative world, the Slavnov–Taylor identities are in presence of θ not strong enough to preserve the form of the action at higher loop order.

3.3 The tree level Green's functions

To be explicit, the non-vanishing tree level Green's functions of our model are at order $n=0$ in θ given by

$$
\begin{aligned}
\Gamma^{(0,0)}_{\bar\psi\psi}(q,p) &= \gamma^\mu p_\mu - m,\\
\Gamma^{(0,0)\mu}_{A\bar\psi\psi}(p,q,r) &= \gamma^\mu,\\
\Gamma^{(0,0)\mu\nu}_{AA}(p,q) &= -\frac{1}{g^2}(p^2 g^{\mu\nu} - p^\mu p^\nu),\\
\Gamma^{(0,0)\mu}_{AB}(p,q) &= -\mathrm{i}p^\mu,\\
\Gamma^{(0,0)}_{BB}(p,q) &= \alpha, \quad \Gamma^{(0,0)}_{\bar c c}(q,p) = p^2,\\
\Gamma^{(0,0)}_{\bar\rho c\psi}(q,p,r) &= \mathrm{i}, \quad \Gamma^{(0,0)}_{\bar\psi c\rho}(q,p,r) = -\mathrm{i},\\
\Gamma^{(0,0)}_{\mu;\sigma c}(q,p) &= -\mathrm{i}p_\mu. \qquad (3.24)
\end{aligned}
$$

It is straightforward to check the tree level ($\ell=0$) Slavnov–Taylor identities (3.18)–(3.23) for $n=0$. The propagators are the bilinear parts of the tree level generating functional of connected Green's functions:

$$
\begin{aligned}
\Delta^{\bar\psi\psi}(q,p) &= -\frac{\gamma^\mu p_\mu + m}{p^2-m^2+\mathrm{i}\epsilon},\\
\Delta^{AA}_{\mu\nu}(p,q) &= \frac{g^2}{p^2+\mathrm{i}\epsilon}\left(g_{\mu\nu} - \left(1-\frac{\alpha}{g^2}\right)\frac{p_\mu p_\nu}{p^2+\mathrm{i}\epsilon}\right),\\
\Delta^{AB}_{\mu}(p,q) &= -\frac{\mathrm{i}p_\mu}{p^2+\mathrm{i}\epsilon}\ ,\\
\Delta^{\bar c c}(q,p) &= -\frac{1}{p^2+\mathrm{i}\epsilon}. \qquad (3.25)
\end{aligned}
$$

At order $n=1$ in θ we have the following tree level 1PI Green's functions:

$$
\Gamma^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r) = -\frac{\mathrm{i}}{2}\theta^{\alpha\beta}p_\alpha r_\beta\gamma^\mu, \qquad (3.26)
$$

$$
\begin{aligned}
\Gamma^{(1,0)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s) &= \mathrm{i}\theta^{\mu\beta}q_\beta\gamma^\nu + \mathrm{i}\theta^{\nu\beta}p_\beta\gamma^\mu\\
&\quad - \frac{\mathrm{i}}{2}\theta^{\mu\nu}(q_\rho - p_\rho)\gamma^\rho, \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{(1,0)\mu\nu\rho}_{AAA}(p,q,r) &= \frac{1}{g^2}\mathrm{i}\theta_{\alpha\beta}\Big(g^{\alpha\mu}g^{\beta\nu}((qr)p^\rho - (pr)q^\rho)\\
&+ g^{\alpha\nu}g^{\beta\rho}((rp)q^\mu - (qp)r^\mu)\\
&+ g^{\alpha\rho}g^{\beta\mu}((pq)r^\nu - (rq)p^\nu)\\
&+ g^{\alpha\mu}((g^{\nu\rho}(pq) - p^\nu q^\rho)r^\beta + (g^{\nu\rho}(rp) - r^\nu p^\rho)q^\beta)\\
&+ g^{\alpha\nu}((g^{\rho\mu}(qr) - q^\rho r^\mu)p^\beta + (g^{\rho\mu}(pq) - p^\rho q^\mu)r^\beta)\\
&+ g^{\alpha\rho}((g^{\mu\nu}(rp) - r^\mu p^\nu)q^\beta + (g^{\mu\nu}(qr) - q^\mu r^\nu)p^\beta)\\
&+ g^{\mu\nu}(p^\rho q^\alpha r^\beta + q^\rho p^\alpha r^\beta) + g^{\nu\rho}(q^\mu r^\alpha p^\beta + r^\mu q^\alpha p^\beta)\\
&+ g^{\rho\mu}(r^\nu p^\alpha q^\beta + p^\nu r^\alpha q^\beta) - g^{\alpha\mu}(g^{\nu\rho}(rq) - r^\nu q^\rho)p^\beta\\
&- g^{\alpha\nu}(g^{\rho\mu}(pr) - p^\rho r^\mu)q^\beta\\
&- g^{\alpha\rho}(g^{\mu\nu}(qp) - q^\mu p^\nu)r^\beta\Big), \qquad (3.28)
\end{aligned}
$$

$$
\Gamma^{(1,0)}_{\bar\rho c\psi}(q,p,r) = \frac{1}{2}\theta^{\alpha\beta}p_\alpha r_\beta, \qquad (3.29)
$$

$$
\Gamma^{(1,0)}_{\bar\psi c\rho}(q,p,r) = \frac{1}{2}\theta^{\alpha\beta}p_\alpha q_\beta\ , \qquad (3.30)
$$

$$
\Gamma^{(1,0)\nu}_{A\bar\rho c\psi}(q,r,p,s) = -\frac{1}{2}\theta^{\nu\beta}p_\beta\ , \qquad (3.31)
$$

$$
\Gamma^{(1,0)\nu}_{A\bar\psi c\rho}(q,r,p,s) = \frac{1}{2}\theta^{\nu\beta}p_\beta. \qquad (3.32)
$$

It is straightforward to check the tree level ($\ell=0$) Slavnov–Taylor identities (3.18)–(3.23) for $n=1$.

3.4 One-loop computation

Using analytic regularisation we compute the one-loop divergent Green's functions up to first order in θ. We choose the Feynman gauge $\alpha=g^2$. At order $n=0$ in θ we find

$$
\begin{aligned}
&\Gamma^{(0,1)}_{\bar\psi\psi}(q,p)\\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\frac{1}{2}N_\psi + 3m\frac{\partial}{\partial m}\right)\Gamma^{(0,0)}_{\bar\psi\psi}(q,p), \qquad (3.33)\\
&\Gamma^{(0,1)\mu}_{A\bar\psi\psi}(p,q,r)\\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\frac{1}{2}N_\psi + 0N_A\right)\Gamma^{(0,0)\mu}_{A\bar\psi\psi}(p,q,r), \qquad (3.34)\\
&\Gamma^{(0,1)\mu\nu}_{AA}(p,q)\\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(-\frac{4}{3}g^2\frac{\partial}{\partial g^2} + 0N_A\right)\Gamma^{(0,0)\mu\nu}_{AA}(p,q), \qquad (3.35)
\end{aligned}
$$

where N_ψ and N_A are the counting operators of electrons $\bar\psi,\psi$ and photons A_μ, respectively. There are no divergences in graphs involving $c,\bar c,B,\sigma^\mu,\rho,\bar\rho$ at order 0 in θ so that $\rho,\bar\rho$ must receive a wave function renormalisation $-(1/2)(\hbar g^2)/((4\pi)^2\varepsilon)N_\rho$ in order to compensate the wave function renormalisation of $\psi,\bar\psi$. The result (3.33)–(3.35) means that at order 0 in θ all one-loop divergences can be removed by a redefinition of the electron wave function, the electron mass and the coupling constant.

At order $n=1$ in θ we find

$$
\Gamma^{(1,1)}_{\bar\psi\psi}(q,p) = 0, \qquad (3.36)
$$

$$
\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r)
$$

$$
\begin{aligned}
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\left(\frac{1}{2}N_\psi + 0N_A\right)\Gamma^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r)\right. && (3.37)\\
&\quad + \mathrm{i}\theta^{\alpha\beta}\left(\frac{1}{2}(p_\nu r_\beta - r_\nu p_\beta)\delta^\mu_\alpha\gamma^\nu - \frac{1}{4}m\delta^\mu_\alpha(2r^\nu + p^\nu)\gamma_{\beta\nu}\right.\\
&\quad \left.\left. - \frac{3}{4}(p^2\delta^\mu_\nu - p^\mu p_\nu)\gamma^\nu_{\alpha\beta} - \frac{3}{2}p_\nu p_\beta\gamma^{\mu\nu}_{\alpha} + \frac{15}{4}m\delta^\mu_\alpha p_\beta\right)\right),
\end{aligned}
$$

$$\Gamma^{(1,1)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s) = \ldots, \qquad (3.38)$$

$$\Gamma^{(1,1)\mu\nu\rho}_{AAA\bar\psi\psi}(p,q,r,s,t) = \ldots, \qquad (3.39)$$

$$\Gamma^{(1,1)}_{\bar\psi\psi\bar\psi\psi}(p,q;r,s) = \frac{\hbar g^2}{(4\pi)^2\varepsilon}\mathrm{i}\theta^{\alpha\beta}\left(\frac{3}{4}g^2\gamma^\mu\otimes\gamma_{\mu\alpha\beta}\right), \qquad (3.40)$$

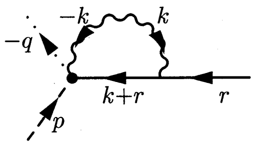
$$\Gamma^{(1,1)\mu\nu}_{AA}(p,q) = 0, \qquad (3.41)$$

$$
\begin{aligned}
&\Gamma^{(1,1)\mu\nu\rho}_{AAA}(p,q,r)\\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(-\frac{4}{3}g^2\frac{\partial}{\partial g^2}\right)\Gamma^{(1,0)\mu\nu\rho}_{AAA}(p,q,r), && (3.42)
\end{aligned}
$$

$$\Gamma^{(1,1)\mu_1\ldots\mu_N}_{A\ldots A}(p_1,\ldots p_N) = \ldots, \quad N\in\{4,5,6\}. \qquad (3.43)$$

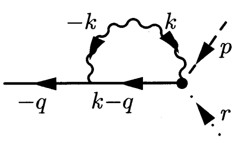
We did not compute the divergent Green's functions $\Gamma^{(1,1)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)$, $\Gamma^{(1,1)\mu\nu\rho}_{AAA\bar\psi\psi}(p,q,r,s,t)$ and $\Gamma^{(1,1)\mu_1\ldots\mu_N}_{A\ldots A}(p_1,\ldots p_N)$ for $N\in\{4,5,6\}$, because they do not give new information for the discussion (see footnote 3 below). The γ-matrices with several indices are antisymmetrised products of γ^μ, such as $\gamma^\nu_\beta = (1/2)(\gamma_\beta\gamma^\nu - \gamma^\nu\gamma_\beta)$. The graphs to compute for the Green's functions (3.36), (3.37) and (3.40)–(3.42) are exactly the same as those given in [22], only the Feynman rules are different. There is no need to print these graphs again. However, there are now divergent graphs involving external fields, which have no analogue in [22]. These graphs are computed to be

$\Gamma^{(1,1)}_{\bar\rho c\psi}(q,p,r) =$



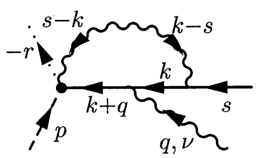
$$
\begin{aligned}
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\\
&\quad\times\left(-\frac{1}{4}r_\beta + \frac{1}{4}\gamma_{\beta\nu}r^\nu - \frac{1}{2}m\gamma_\beta\right), && (3.44)
\end{aligned}
$$

$\Gamma^{(1,1)}_{\bar\psi c\rho}(q,p,r) =$



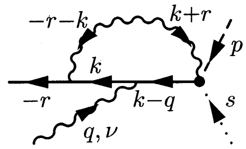
$$
\begin{aligned}
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\\
&\quad\times\left(-\frac{1}{4}q_\beta - \frac{1}{4}\gamma_{\beta\nu}q^\nu + \frac{1}{2}m\gamma_\beta\right), && (3.45)
\end{aligned}
$$

$\Gamma^{(1,1)\nu}_{A\bar\rho c\psi}(q,r,p,s) =$



$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\left(-\frac{1}{4}\delta^\nu_\beta + \frac{1}{4}\gamma^\nu_\beta\right), \qquad (3.46)$$

$\Gamma^{(1,1)\nu}_{A\bar\psi c\rho}(q,r,p,s) =$



$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\left(\frac{1}{4}\delta^\nu_\beta + \frac{1}{4}\gamma^\nu_\beta\right). \qquad (3.47)$$

The external fields $\bar\rho,\rho$ are symbolised by dotted lines and the ghost c by dashed lines, everything else is as in [22]. A vertex with a dot is of first order in θ.

First, the $(n=1,\ell=1)$ Slavnov–Taylor identities (3.18) and (3.20)–(3.23) are fulfilled, as already expected from general considerations. For us the importance of these identities consists in testing the graph computations performed by a Mathematica$^{\mathrm{TM}}$ program [28]. Next, the one-loop divergent Green's functions at first order in θ are considerably *different* from their tree level form. The question is then how many of these divergences can be removed by a field redefinition.

3.5 Field redefinitions

A field redefinition $\mathcal{F}$ must preserve the Slavnov–Taylor identity, hence we have to require

$$
\begin{aligned}
&\mathcal{F} = \left(\exp\left(\sum_i\int\Psi_i\,\frac{\delta}{\delta\Phi_i}\right)\Bigg|_{\Phi_j=0,\delta\Psi_j/\delta\Phi_i=0}\right)_{\Psi_i=\Psi_i[\Phi_j]},\\
&\mathcal{S}(\mathcal{F}\Gamma) = 0, && (3.48)
\end{aligned}
$$

where the functional $\Psi_i[\Phi_j]$ of the fields Φ_j must be of the same power-counting dimension, ghost charge and hermiticity as the field Φ_i. We make the ansatz

$$
\begin{aligned}
\mathcal{F}\psi &= \psi - \frac{1}{2}\tau\theta^{\alpha\beta}A_\alpha\partial_\beta\psi + \frac{\mathrm{i}}{4}\tau\theta^{\alpha\beta}mA_\alpha\gamma_\beta\psi\\
&\quad + \frac{3}{8}\tau'\theta^{\alpha\beta}F^{\mu\nu}\gamma_{\mu\nu\alpha\beta}\psi, && (3.49)\\
\mathcal{F}\bar\psi &= \bar\psi - \frac{1}{2}\tau\theta^{\alpha\beta}\partial_\beta\bar\psi A_\alpha - \frac{\mathrm{i}}{4}\tau\theta^{\alpha\beta}\bar\psi\gamma_\beta mA_\alpha\\
&\quad + \frac{3}{8}\tau'\theta^{\alpha\beta}\bar\psi F^{\mu\nu}\gamma_{\mu\nu\alpha\beta}, && (3.50)\\
\mathcal{F}\rho &= \rho - \frac{1}{2}\tau\theta^{\alpha\beta}\partial_\beta(A_\alpha\rho) + \frac{\mathrm{i}}{4}\tau\theta^{\alpha\beta}mA_\alpha\gamma_\beta\rho\\
&\quad - \frac{3}{8}\tau'\theta^{\alpha\beta}F^{\mu\nu}\gamma_{\mu\nu\alpha\beta}\rho, && (3.51)\\
\mathcal{F}\bar\rho &= \bar\rho - \frac{1}{2}\tau\theta^{\alpha\beta}\partial_\beta(\bar\rho A_\alpha) - \frac{\mathrm{i}}{4}\tau\theta^{\alpha\beta}\bar\rho\gamma_\beta mA_\alpha\\
&\quad - \frac{3}{8}\tau'\theta^{\alpha\beta}\bar\rho F^{\mu\nu}\gamma_{\mu\nu\alpha\beta}, && (3.52)\\
\mathcal{F}\sigma^\mu &= \sigma^\mu + \theta^{\mu\beta}\bar\rho\left(\frac{1}{4}\tau(\delta^\nu_\beta - \gamma^\nu_\beta)(\partial_\nu\psi - \mathrm{i}A_\nu\psi)\right.\\
&\quad \left. + \frac{\mathrm{i}}{2}\tau A_\beta\psi - \frac{\mathrm{i}}{2}\tau\gamma_\beta m\psi\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \theta^{\mu\beta}\bigg(\frac{1}{4}\tau(\partial_\nu\bar\psi + \mathrm{i}\bar\psi A_\nu)(\delta^\nu_\beta + \gamma^\nu_\beta) \\
&- \frac{\mathrm{i}}{2}\tau\bar\psi A_\beta + \frac{\mathrm{i}}{2}\tau\bar\psi\gamma_\beta m\bigg)\rho, \qquad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}A_\mu = A_\mu, \quad \mathcal{F}c = c, \quad \mathcal{F}\kappa = \kappa, \\
&\mathcal{F}\bar c = \bar c, \quad \mathcal{F}B = B, \qquad (3.54)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\mathcal{F}(\Gamma^{(0,0)}) = \Gamma^{(0,0)} \\
&+ \tau\theta^{\alpha\beta}\bigg(-\frac{1}{2}\bar\psi\mathrm{i}\gamma^\mu\partial_\mu A_\alpha\partial_\beta\psi + \frac{1}{2}\bar\psi\gamma^\mu A_\alpha\partial_\beta A_\mu\psi \\
&- \frac{1}{4}\bar\psi\mathrm{i}\gamma^\mu F_{\alpha\beta}D_\mu\psi + \frac{3}{8}\bar\psi m F_{\alpha\beta}\psi \\
&+ \frac{1}{4}\bar\psi m\gamma^\mu_\beta(2A_\alpha D_\mu\psi + \partial_\mu A_\alpha\psi)\bigg) \\
&+ \frac{3}{4}\tau'\theta^{\alpha\beta}(-\mathrm{i}\bar\psi\gamma_{\mu\nu\alpha}\partial_\beta F^{\mu\nu}\psi + \mathrm{i}\bar\psi\gamma_{\mu\alpha\beta}\partial_\nu F^{\nu\mu}\psi \\
&- \bar\psi m\gamma_{\mu\nu\alpha\beta}F^{\mu\nu}\psi) \\
&+ \tau\theta^{\alpha\beta}\bar\rho\partial_\alpha c\bigg(\frac{1}{4}(\delta^\nu_\beta - \gamma^\nu_\beta)(\partial_\nu\psi - \mathrm{i}A_\nu\psi) - \frac{\mathrm{i}}{2}\gamma_\beta m\psi\bigg) \\
&+ \tau\theta^{\alpha\beta}\bigg(\frac{1}{4}(\partial_\nu\bar\psi + \mathrm{i}\bar\psi A_\nu)(\delta^\nu_\beta + \gamma^\nu_\beta) + \frac{\mathrm{i}}{2}\bar\psi\gamma_\beta m\bigg)\partial_\alpha c\rho \\
&+ \mathcal{O}(\theta^2). \qquad (3.55)
\end{aligned}
$$

The corresponding Green's functions are

$$
\begin{aligned}
&(\mathcal{F}\Gamma)^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r) \\
&= \mathrm{i}\tau\theta^{\mu\beta}\bigg(-\frac{1}{2}(r_\nu p_\beta - p_\nu r_\beta)\gamma^\nu + \frac{3}{4}mp_\beta \\
&- \frac{1}{4}m(2r^\nu + p^\nu)\gamma_{\beta\nu}\bigg) \\
&+ \frac{3}{4}\mathrm{i}\tau'\theta^{\alpha\beta}\Big(-2p_\beta p_\nu\gamma^{\mu\nu}{}_\alpha - \gamma_{\nu\alpha\beta}(p^2 g^{\mu\nu} - p^\mu p^\nu) \\
&- 2mp_\nu\gamma^{\mu\nu}{}_{\alpha\beta}\Big), \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
&(\mathcal{F}\Gamma)^{(1,0)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s) \\
&= \mathrm{i}\tau\theta^{\alpha\beta}\bigg(-\frac{1}{2}(p_\beta + q_\beta)(\delta^\mu_\alpha\gamma^\nu + \delta^\nu_\alpha\gamma^\mu) \\
&- \frac{1}{2}m(\delta^\mu_\alpha\gamma^\nu_\beta + \delta^\mu_\alpha\gamma^\mu_\beta)\bigg), \qquad (3.57)
\end{aligned}
$$

$$
\begin{aligned}
&(\mathcal{F}\Gamma)^{(1,0)}_{\bar\rho c\psi}(q,p,r) \\
&= \tau\theta^{\alpha\beta}p_\alpha\bigg(-\frac{1}{4}(r_\beta - \gamma^\nu_\beta r_\nu) - \frac{1}{2}m\gamma_\beta\bigg), \qquad (3.58)
\end{aligned}
$$

$$
\begin{aligned}
&(\mathcal{F}\Gamma)^{(1,0)}_{\bar\psi c\rho}(q,p,r) \\
&= \tau\theta^{\alpha\beta}p_\alpha\bigg(-\frac{1}{4}(q_\beta + \gamma^\nu_\beta q_\nu) + \frac{1}{2}m\gamma_\beta\bigg), \qquad (3.59)
\end{aligned}
$$

$$
(\mathcal{F}\Gamma)^{(1,0)\nu}_{A\bar\rho c\psi}(q,r,p,s) = \tau\theta^{\alpha\beta}p_\alpha\bigg(-\frac{1}{4}(\delta^\nu_\beta - \gamma^\nu_\beta)\bigg), \qquad (3.60)
$$

$$
(\mathcal{F}\Gamma)^{(1,0)\nu}_{A\bar\psi c\rho}(q,r,p,s) = \tau\theta^{\alpha\beta}p_\alpha\bigg(\frac{1}{4}(\delta^\nu_\beta + \gamma^\nu_\beta)\bigg). \qquad (3.61)
$$

The Slavnov–Taylor identities (3.18)–(3.23) are verified. Now (3.37) and (3.44)–(3.47) can be rewritten as

$$
\begin{aligned}
&\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r) = \frac{\hbar g^2}{(4\pi)^2\varepsilon} \\
&\times\bigg(\bigg(\frac{1}{2}N_\psi + 0N_A\bigg)\Gamma^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r) \\
&+ \bigg(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\tau'}\bigg)(\mathcal{F}\Gamma)^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r) \\
&+ \mathrm{i}\theta^{\alpha\beta}\bigg(3m\delta^\mu_\alpha p_\beta + \frac{3}{2}mp_\nu\gamma^{\mu\nu}{}_{\alpha\beta}\bigg)\bigg), \qquad (3.62)
\end{aligned}
$$

$$
\Gamma^{(1,1)}_{\text{ext.field}} = \frac{\hbar g^2}{(4\pi)^2\varepsilon}\frac{\partial}{\partial\tau}(\mathcal{F}\Gamma)^{(1,0)}_{\text{ext.field}}, \qquad (3.63)
$$

where $_{\text{ext.field}}$ stands for $_{\bar\rho c\psi}(q,p,r)$, $_{\bar\psi c\rho}(q,p,r)$, $^{\nu}_{A\bar\rho c\psi}(q,r,p,s)$ and $^{\nu}_{A\bar\psi c\rho}(q,r,p,s)$. In other words, the one-loop divergences in the Green's functions involving external fields and, for $m=0$, in $\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r)$ can be removed by field redefinitions. Due to the Slavnov–Taylor identity these field redefinitions remove all one-loop divergences in $\Gamma^{(1,1)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)$ and $\Gamma^{(1,1)\mu\nu\rho}_{AAA\bar\psi\psi}(p,q,r,s,t)$ as well, and $\Gamma^{(1,1)\mu_1\dots\mu_N}_{A\dots A}(p_1,\dots p_N)$ is convergent for $N\in\{4,5,6\}$[3]. There remain only the divergence in the electron four-point function (3.40) and the two mass terms in (3.62). It is remarkable that these remaining divergences coincide exactly (with the same numerical coefficients!) with the result obtained in [22] where the electrons are Seiberg–Witten expanded! Moreover, there are no divergences in the photon N-point functions $\Gamma^{(1,1)\mu_1,\dots\mu_N}_{A\dots A}(p_1,\dots,p_N)$ after *the same* renormalisation of the coupling constant as in QED, see (3.35). Again, this coincides with the results found in [22] where the fermions are Seiberg–Witten expanded as well. This is a remarkable result: The physical (i.e. modulo field redefinitions) one-loop divergences are insensitive for the choice of non-commutative or Seiberg–Witten expanded electrons in θ-expanded non-commutative QED.

[3] Since all divergences in Green's functions involving external fields are removed by a field redefinition, see (3.63), the $(n=1,\ell=1)$ Slavnov–Taylor identity (3.19) implies that the divergent part of $\Gamma^{(1,1)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)$ is transversal (contraction with p_μ yields zero) after the field redefinitions (3.49)–(3.54), because the remaining divergences in $\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r)$ are independent of r. Since $\Gamma^{(1,1)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)$ is linear in momentum variables and symmetric under $(p,\mu)\leftrightarrow(q,\nu)$, it must be zero. In the same way one proves $\Gamma^{(1,1)\mu\nu\rho}_{AAA\bar\psi\psi}(p,q,r,s,t)=0$ after the field redefinitions (3.49)–(3.54). Similarly, the photon N-point functions $\Gamma^{(1,1)\mu_1\dots\mu_N}_{A\dots A}(p_1,\dots p_N)$ for $N\in\{4,5,6\}$ are transversal in all momenta, but because they are at most quadratic (for $N=4$) in the momentum variables, they must vanish. This short proof shows that the computation of (3.38), (3.39) and (3.43) was not necessary

4 Expanding the action (Case II)

In this section we complete the first order analysis of the Seiberg–Witten map on quantum level by leaving it out completely: We repeat the analysis of the previous section without applying the Seiberg–Witten map to the bosonic sector. The result is a "tilted" BRST symmetry in both bosonic and fermionic sectors leading to a tower of symmetries involving both bosonic and fermionic actions.

4.1 Classical analysis

The expansion of the action (2.9), including the ghost sector, is now performed according to

$$(f \star g)(x) = f(x)g(x) + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\partial_\alpha f(x)\partial_\beta g(x) + \mathcal{O}(\theta^2),$$
$$\hat{\Phi} = \Phi, \quad \forall \hat{\Phi} \in \{\hat{A}_\mu, \hat{\psi}, \hat{\bar{\psi}}, \hat{c}, \hat{\bar{c}}, \hat{B}, \hat{\bar{\rho}}, \hat{\rho}, \hat{\sigma}^\mu, \hat{\kappa}\}. \quad (4.1)$$

This leads to the expanded action

$$\Sigma^{\{n\}}_{\theta-\exp} = \sum_{i=0}^{n} \Sigma^{(i)}, \quad (4.2)$$

which up to first order in θ (in which we are only interested for now) reads

$$\Sigma^{(0)}_{\mathrm{cl}} = \int \mathrm{d}^4x \left(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(\mathrm{i}\gamma^\mu D_\mu - m\right)\psi \right), \quad (4.3)$$

$$\Sigma^{(1)}_{\mathrm{cl}} = \int \mathrm{d}^4x \left(-\frac{1}{2g^2}\theta^{\alpha\beta}F_{\mu\nu}\partial_\alpha A^\mu \partial_\beta A^\nu + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\bar{\psi}\gamma^\mu\partial_\alpha A_\mu \partial_\beta\psi \right), \quad (4.4)$$

$$\Sigma^{(0)}_{\mathrm{gf}} = \int \mathrm{d}^4x \left(B\partial^\mu A_\mu - \bar{c}\partial^\mu\partial_\mu c + \frac{\alpha}{2}BB \right), \quad (4.5)$$

$$\Sigma^{(1)}_{\mathrm{gf}} = \int \mathrm{d}^4x (\theta^{\alpha\beta}\partial^\mu\bar{c}\,\partial_\alpha A_\mu\,\partial_\beta c). \quad (4.6)$$

We expand (2.4) using (4.1) to first order in θ:

$$\begin{aligned}
s^{(0)}A_\mu &= \partial_\mu c, \quad s^{(0)}c = 0,\\
s^{(0)}\psi &= \mathrm{i}c\psi, \quad s^{(0)}\bar{\psi} = -\mathrm{i}\bar{\psi}c,\\
s^{(1)}A_\mu &= \theta^{\alpha\beta}\partial_\alpha A_\mu \partial_\beta c,\\
s^{(1)}c &= -\frac{1}{2}\theta^{\alpha\beta}(\partial_\alpha c)(\partial_\beta c),\\
s^{(1)}\psi &= -\frac{1}{2}\theta^{\alpha\beta}\partial_\alpha c\partial_\beta\psi,\\
s^{(1)}\bar{\psi} &= -\frac{1}{2}\theta^{\alpha\beta}\partial_\beta\bar{\psi}\partial_\alpha c. \quad (4.7)
\end{aligned}$$

The above transformations are θ-graded in both bosonic and fermionic sectors. The θ-expanded BRST transformations (4.7) fulfil (3.10) and (3.11). Again we also expand the term with external fields leading to (3.13) with the BRST transformations defined in (4.7). The full tree level generating functional is defined by (3.14), now with the classical and gauge-fixing actions given by (4.6).

Again the full set of BRST symmetries must be expressed by Slavnov–Taylor identities (3.16) and (3.18)–(3.23).

4.2 The tree level Green's functions

At order $n = 0$ in θ the tree level Green's functions of Case II are clearly the same as before (3.24). At order $n = 1$ in θ we now have the following non-vanishing tree level Green's functions:

$$\Gamma^{(1,0)\mu}_{A\bar{\psi}\psi}(p,q,r,s) = -\frac{\mathrm{i}}{2}\theta^{\alpha\beta}p_\alpha r_\beta\gamma^\mu, \quad (4.8)$$
$$\Gamma^{(1,0)\mu}_{\bar{c}Ac}(p,q,r) = \mathrm{i}\theta^{\alpha\beta}p^\mu q_\alpha r_\beta, \quad (4.9)$$
$$\begin{aligned}\Gamma^{(1,0)\mu\nu\rho}_{AAA}(p,q,r) = \frac{\mathrm{i}}{g^2}\theta^{\alpha\beta}\Big(&(g^{\mu\nu}p^\rho - g^{\rho\mu}p^\nu)q_\alpha r_\beta\\
&+ (g^{\nu\rho}q^\mu - g^{\mu\nu}q^\rho)r_\alpha p_\beta\\
&+ (g^{\rho\mu}r^\nu - g^{\nu\rho}r^\mu)p_\alpha q_\beta\Big), \quad (4.10)\end{aligned}$$
$$\Gamma^{(1,0)\nu}_{\mu;A\sigma c}(q,r,p) = \delta^\nu_\mu\theta^{\alpha\beta}p_\alpha q_\beta, \quad (4.11)$$
$$\Gamma^{(1,0)}_{\bar{\rho}c\psi}(q,p,r) = \frac{1}{2}\theta^{\alpha\beta}p_\alpha r_\beta, \quad (4.12)$$
$$\Gamma^{(1,0)}_{\bar{\psi}c\rho}(q,p,r) = \frac{1}{2}\theta^{\alpha\beta}p_\alpha q_\beta, \quad (4.13)$$
$$\Gamma^{(1,0)}_{\kappa cc}(p,q,r) = \theta^{\alpha\beta}q_\alpha r_\beta. \quad (4.14)$$

It is straightforward to check the $(n = 1, \ell = 0)$ Slavnov–Taylor identities (3.18)–(3.23).

4.3 One-loop computation

The one-loop results for order $n = 0$ in θ are the same as before (3.33)–(3.35). At order $n = 1$ in θ we find the following divergent Green's functions in analytic regularisation (using again the Feynman gauge $\alpha = g^2$):

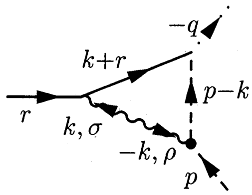
$$\Gamma^{(1,1)}_{\bar{\psi}\psi}(q,p) = 0, \quad (4.15)$$

$$\begin{aligned}&\Gamma^{(1,1)\mu}_{A\bar{\psi}\psi}(p,q,r)\\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\left(\frac{1}{2}N_\psi + 0N_A\right)\Gamma^{(1,0)\mu}_{A\bar{\psi}\psi}(p,q,r)\right.\\
&+ \mathrm{i}\theta^{\alpha\beta}\left(-\frac{1}{2}p_\alpha r_\beta\gamma^\mu - \frac{1}{4}(2r^\mu + p^\mu)p_\beta\gamma_\alpha\right. \quad (4.16)\\
&\left.\left. - \frac{1}{4}\delta^\mu_\alpha(2r_\nu + p_\nu)p_\beta\gamma^\nu - \frac{5}{4}p_\nu p_\beta\gamma^{\mu\nu}_{\ \ \alpha} + \frac{7}{2}m\delta^\mu_\alpha p_\beta\right)\right),\end{aligned}$$

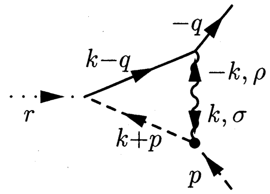
$$\Gamma^{(1,1)\mu\nu}_{AA\bar{\psi}\psi}(p,q,r,s) = \ldots, \quad (4.17)$$
$$\Gamma^{(1,1)\mu\nu\rho}_{AAA\bar{\psi}\psi}(p,q,r,s,t) = \ldots, \quad (4.18)$$
$$\Gamma^{(1,1)}_{\bar{\psi}\psi;\bar{\psi}\psi}(p,q,r,s) = 0, \quad (4.19)$$
$$\Gamma^{(1,1)\mu\nu}_{AA}(p,q) = 0, \quad (4.20)$$

$$\Gamma^{(1,1)\mu\nu\rho}_{AAA}(p,q,r) = \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(-\frac{4}{3}g^2\frac{\partial}{\partial g^2}+0N_A\right)\Gamma^{(1,0)\mu\nu\rho}_{AAA}(p,q,r), \tag{4.21}$$

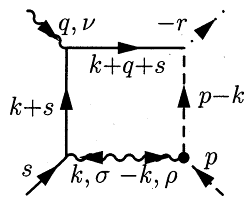
$$\Gamma^{(1,1)\mu_1\dots\mu_N}_{A\dots A}(p_1,\dots p_N) = \dots, \quad N\in\{4,5,6\}, \tag{4.22}$$

$\Gamma^{(1,1)}_{\bar\rho c\psi}(q,p,r) =$ 

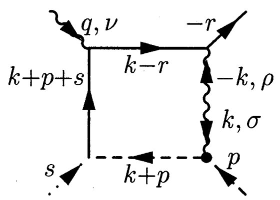
$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\left(-\frac{1}{4}q_\beta-\frac{1}{4}m\gamma_\beta-\frac{1}{4}\gamma_{\beta\nu}q^\nu\right), \tag{4.23}$$

$\Gamma^{(1,1)}_{\bar\psi c\rho}(q,p,r) =$ 

$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\left(-\frac{1}{4}r_\beta+\frac{1}{4}m\gamma_\beta+\frac{1}{4}\gamma_{\beta\nu}r^\nu\right), \tag{4.24}$$

$\Gamma^{(1,1)\nu}_{A\bar\rho c\psi}(q,r,p,s) =$ 

$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\left(\frac{1}{4}\delta^\nu_\beta+\frac{1}{4}\gamma^\nu_\beta\right), \tag{4.25}$$

$\Gamma^{(1,1)\nu}_{A\bar\psi c\rho}(q,r,p,s) =$ 

$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}p_\alpha\left(-\frac{1}{4}\delta^\nu_\beta+\frac{1}{4}\gamma^\nu_\beta\right). \tag{4.26}$$

The $(n=1,\ell=1)$ Slavnov–Taylor identities (3.18) and (3.20)–(3.23) are verified.

4.4 Field redefinitions

We try again to absorb the divergences in (4.23)–(4.26) and most of (4.16) by field redefinitions. We make the ansatz

$$\begin{aligned}\mathcal{F}\psi &= \psi \\ &+\theta^{\alpha\beta}\left(-\frac{1}{4}\tau\gamma^\nu_\alpha\partial_\beta A_\nu\psi+\frac{3}{8}\tau'\gamma_{\mu\nu\alpha\beta}F^{\mu\nu}\psi-\frac{1}{8}\tau''F_{\alpha\beta}\psi\right),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\mathcal{F}\bar\psi &= \bar\psi \\ &+\theta^{\alpha\beta}\left(\frac{1}{4}\tau\bar\psi\gamma^\nu_\alpha\partial_\beta A_\nu+\frac{3}{8}\tau'\bar\psi\gamma_{\mu\nu\alpha\beta}F^{\mu\nu}-\frac{1}{8}\tau''\bar\psi F_{\alpha\beta}\right),\end{aligned} \tag{4.28}$$

$$\begin{aligned}\mathcal{F}\bar\rho &= \bar\rho \\ &+\theta^{\alpha\beta}\left(\frac{1}{4}\tau\bar\rho\gamma^\nu_\alpha\partial_\beta A_\nu-\frac{3}{8}\bar\rho\tau'\gamma_{\mu\nu\alpha\beta}F^{\mu\nu}+\frac{1}{8}\tau''\bar\rho F_{\alpha\beta}\right),\end{aligned} \tag{4.29}$$

$$\begin{aligned}\mathcal{F}\rho &= \rho \\ &+\theta^{\alpha\beta}\left(-\frac{1}{4}\tau\gamma^\nu_\alpha\partial_\beta A_\nu\rho-\frac{3}{8}\tau'\gamma_{\mu\nu\alpha\beta}F^{\mu\nu}\rho+\frac{1}{8}\tau''F_{\alpha\beta}\rho\right),\end{aligned} \tag{4.30}$$

$$\mathcal{F}A_\mu = A_\mu-\frac{3}{4}\mathrm{i}g^2\tau'''\theta^{\alpha\beta}\bar\psi\gamma_{\mu\alpha\beta}\psi, \tag{4.31}$$

$$\begin{aligned}\mathcal{F}\sigma^\mu &= \sigma^\mu \\ &+\frac{1}{4}\tau\theta^{\mu\beta}((\partial_\nu\bar\rho+\mathrm{i}\bar\rho A_\nu)(\delta^\nu_\beta+\gamma^\nu_\beta)\psi-\mathrm{i}\bar\rho m\gamma_\beta\psi \\ &-\bar\psi(\delta^\nu_\beta-\gamma^\nu_\beta)(\partial_\nu\rho-\mathrm{i}A_\nu\rho)-\mathrm{i}\bar\psi m\gamma_\beta\rho),\end{aligned} \tag{4.32}$$

$$\mathcal{F}c=c,\quad \mathcal{F}\kappa=\kappa,\quad \mathcal{F}\bar c=\bar c,\quad \mathcal{F}B=B, \tag{4.33}$$

which gives

$$\begin{aligned}\mathcal{F}(\Gamma^{(0,0)}) &= \Gamma^{(0,0)} \\ &+\tau\theta^{\alpha\beta}\left(-\frac{1}{2}\bar\psi\mathrm{i}\gamma^\nu\partial_\beta A_\nu\partial_\alpha\psi+\frac{1}{4}\bar\psi\mathrm{i}\gamma_\alpha\partial^\nu\partial_\beta A_\nu\psi\right. \\ &+\frac{1}{2}\bar\psi\mathrm{i}\gamma_\alpha\partial_\beta A_\nu\partial^\nu\psi+\frac{1}{4}\bar\psi\mathrm{i}\gamma^{\mu\nu}_{\ \ \alpha}\partial_\mu\partial_\beta A_\nu\psi \\ &\left.-\frac{1}{2}\bar\psi\gamma^\nu A_\alpha\partial_\beta A_\nu\psi+\frac{1}{2}\bar\psi\gamma_\alpha A^\nu\partial_\beta A_\nu\psi\right) \\ &+\frac{3}{4}\tau'\theta^{\alpha\beta}\Big(-\mathrm{i}\bar\psi\gamma_{\mu\nu\alpha}\partial_\beta F^{\mu\nu}\psi+\mathrm{i}\bar\psi\gamma_{\mu\alpha\beta}\partial_\nu F^{\nu\mu}\psi \\ &-\bar\psi m\gamma_{\mu\nu\alpha\beta}F^{\mu\nu}\psi\Big) \\ &+\tau''\theta^{\alpha\beta}\left(-\frac{1}{8}\bar\psi\mathrm{i}\gamma^\mu(\partial_\mu F_{\alpha\beta}\psi+2F_{\alpha\beta}D_\mu\psi)\right. \\ &\left.+\frac{1}{4}\bar\psi mF_{\alpha\beta}\psi\right) \\ &-\frac{3}{4}\tau'''\theta^{\alpha\beta}\Big(\bar\psi\mathrm{i}\gamma_{\mu\alpha\beta}(\partial_\nu F^{\nu\mu}-g^2\partial^\mu B)\psi \\ &+g^2(\bar\psi\gamma^\mu\psi)(\bar\psi\mathrm{i}\gamma_{\mu\alpha\beta}\psi)\Big) \\ &+\frac{1}{4}\tau\theta^{\alpha\beta}\Big((\partial_\nu\bar\rho+\mathrm{i}\bar\rho A_\nu)(\delta^\nu_\beta+\gamma^\nu_\beta)\partial_\alpha c\psi-\mathrm{i}\bar\rho m\gamma_\beta\partial_\alpha c\psi \\ &+\bar\psi\partial_\alpha c(\delta^\nu_\beta-\gamma^\nu_\beta)(\partial_\nu\rho-\mathrm{i}A_\nu\rho)+\mathrm{i}\bar\psi m\partial_\alpha c\gamma_\beta\rho\Big) \\ &+\mathcal{O}(\theta^2).\end{aligned} \tag{4.34}$$

The corresponding Green's functions are

$$\begin{aligned}(\mathcal{F}\Gamma)^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r) &= \mathrm{i}\theta^{\alpha\beta}\left(-\frac{1}{2}\tau p_\alpha r_\beta\gamma^\mu-\frac{1}{4}\tau(p^\mu+2r^\mu)p_\beta\gamma_\alpha\right. \\ &-\frac{1}{4}\tau''(p_\nu+2r_\nu)p_\beta\delta^\mu_\alpha\gamma^\nu+\left(\frac{1}{4}\tau-\frac{3}{2}\tau'\right)p_\nu p_\beta\gamma^{\mu\nu}_{\ \ \alpha} \\ &+\frac{3}{4}(\tau'''-\tau')(p^2g^{\mu\nu}-p^\mu p^\nu)\gamma_{\nu\alpha\beta} \\ &\left.+\frac{1}{2}\tau''mp_\beta\delta^\mu_\alpha-\frac{3}{2}\tau'mp^\nu\gamma^{\mu\nu}_{\ \ \alpha\beta}\right),\end{aligned} \tag{4.35}$$

$(\mathcal{F}\Gamma)^{(1,0)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)$

$$= \mathrm{i}\theta^{\alpha\beta}\left(\frac{1}{2}(\tau q_\beta - \tau'' p_\beta)\delta^\mu_\alpha\gamma^\nu + \frac{1}{2}(\tau p_\beta - \tau'' q_\beta)\delta^\nu_\alpha\gamma^\mu - \frac{1}{2}\tau(p_\beta+q_\beta)g^{\mu\nu}\gamma_\alpha\right), \tag{4.36}$$

$$(\mathcal{F}\Gamma)^{(1,0)}_{\bar\psi\psi;\bar\psi\psi}(p,q,r,s) = -\frac{3}{4}\tau'''\mathrm{i}\theta^{\alpha\beta}g^2\gamma^\mu\otimes\gamma_{\mu\alpha\beta}, \tag{4.37}$$

$$(\mathcal{F}\Gamma)^{(1,0)}_{B\bar\psi\psi}(p,q,r) = -\frac{3}{4}\tau'''\mathrm{i}\theta^{\alpha\beta}g^2p^\mu\gamma_{\mu\alpha\beta}, \tag{4.38}$$

$$(\mathcal{F}\Gamma)^{(1,0)}_{\bar\rho c\psi}(q,p,r) = \frac{1}{4}\tau\theta^{\alpha\beta}p_\alpha(q_\nu(-\delta^\nu_\beta-\gamma^\nu_\beta)-m\gamma_\beta), \tag{4.39}$$

$$(\mathcal{F}\Gamma)^{(1,0)}_{\bar\psi c\rho}(q,p,r) = \frac{1}{4}\tau\theta^{\alpha\beta}p_\alpha(r_\nu(-\delta^\nu_\beta+\gamma^\nu_\beta)+m\gamma_\beta), \tag{4.40}$$

$$(\mathcal{F}\Gamma)^{(1,0)\nu}_{A\bar\rho c\psi}(q,r,p,s) = \frac{1}{4}\tau\theta^{\alpha\beta}p_\alpha q_\nu(\delta^\nu_\beta+\gamma^\nu_\beta), \tag{4.41}$$

$$(\mathcal{F}\Gamma)^{(1,0)}_{A\bar\psi c\rho}(q,r,p,s) = \frac{1}{4}\tau\theta^{\alpha\beta}p_\alpha q_\nu(-\delta^\nu_\beta+\gamma^\nu_\beta). \tag{4.42}$$

The $(n=1,\ell=0)$ Slavnov–Taylor identities (3.18)–(3.23) are verified. Now (4.16), (4.19) and (4.23)–(4.26) can be rewritten as

$$\begin{aligned}\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r) &= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\left(\frac{1}{2}N_\psi + 0N_A\right)\Gamma^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r)\right.\\ &+ \left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\tau'}+\frac{\partial}{\partial\tau''}+\frac{\partial}{\partial\tau'''}\right)(\mathcal{F}\Gamma)^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r)\\ &\left.+\ \mathrm{i}\theta^{\alpha\beta}\left(3m\delta^\mu_\alpha p_\beta + \frac{3}{2}mp_\nu\gamma^{\mu\nu}{}_{\alpha\beta}\right)\right),\end{aligned} \tag{4.43}$$

$$\Gamma^{(1,1)}_{\bar\psi\psi;\bar\psi\psi}(p,q,r,s) = \frac{\hbar g^2}{(4\pi)^2\varepsilon} \times\left(\frac{\partial}{\partial\tau'''}(\mathcal{F}\Gamma)^{(1,0)}_{\bar\psi\psi;\bar\psi\psi}(p,q,r,s) + \frac{3}{4}\mathrm{i}\theta^{\alpha\beta}g^2\gamma^\mu\otimes\gamma_{\mu\alpha\beta}\right), \tag{4.44}$$

$$\Gamma^{(1,1)}_{B\bar\psi\psi}(p,q,r) = \frac{\hbar g^2}{(4\pi)^2\varepsilon} \times\left(\frac{\partial}{\partial\tau'''}(\mathcal{F}\Gamma)^{(1,0)}_{B\bar\psi\psi}(p,q,r) + \frac{3}{4}\mathrm{i}\theta^{\alpha\beta}g^2p^\mu\gamma_{\mu\alpha\beta}\right), \tag{4.45}$$

$$\Gamma^{(1,1)}_{\text{ext.field}} = \frac{\hbar g^2}{(4\pi)^2\varepsilon}\frac{\partial}{\partial\tau}(\mathcal{F}\Gamma)^{(1,0)}_{\text{ext.field}}, \tag{4.46}$$

where $_{\text{ext.field}}$ stands for $_{\bar\rho c\psi}(q,p,r)$, $_{\bar\psi c\rho}(q,p,r)$, $^\nu_{A\bar\rho c\psi}(q,r,p,s)$ and $^\nu_{A\bar\psi c\rho}(q,r,p,s)$. Thus, the result after field redefinitions is the same as in Case I and [22], provided that a $\hbar$ renormalisation of the tree level gauge-fixing action $\Sigma^{(0)}_{\text{gf}}$ from (4.5) to

$$\Sigma'^{(0)}_{\text{gf}} = \int \mathrm{d}^4x\left(B\partial^\mu\left(A_\mu - \frac{3}{4}g^2\frac{\hbar g^2}{(4\pi)^2\varepsilon}\theta^{\alpha\beta}\bar\psi\gamma_{\mu\alpha\beta}\psi\right) - \bar c\partial^\mu\partial_\mu c + \frac{\alpha}{2}BB\right) \tag{4.47}$$

is performed. In summary, up to field redefinitions the one-loop computations of Green's functions up to first order in θ are completely independent of the application of the Seiberg–Witten map

(1) to both electrons and photons [22],
(2) to photons only, Case I, or
(3) to neither photons nor electrons, Case II.

In the next section we shall explain why this has to be the case.

First let us point out a possibility which we have overlooked in [22] and which becomes apparent from the loop calculation of Case II. Putting $\tau' = \tau''' = 0$ in (4.34) we have instead of (4.43) and (4.44)

$$\begin{aligned}\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r) &= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\left(\frac{1}{2}N_\psi + 0N_A\right)\Gamma^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r)\right.\\ &+ \left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\tau''}\right)(\mathcal{F}\Gamma)^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r)\\ &\left.+\ \mathrm{i}\theta^{\alpha\beta}\left(3m\delta^\mu_\alpha p_\beta - \frac{3}{2}p_\nu p_\beta\gamma^{\mu\nu}{}_\alpha\right)\right),\end{aligned} \tag{4.48}$$

$$\Gamma^{(1,1)}_{\bar\psi\psi;\bar\psi\psi}(p,q,r,s) = 0. \tag{4.49}$$

The same result can obviously be achieved for the treatments of [22] and Case I as well. This is the minimal field redefinition in the sense that only two non-absorbable one-loop divergences remain. It is tempting to try an extended non-commutative initial action

$$\begin{aligned}\hat\Sigma^e_{\text{cl}} &= \hat\Sigma_{\text{cl}} + g_e\int \mathrm{d}^4x\mathrm{i}\theta^{\alpha\beta}\hat{\bar\psi}\star\gamma^{\mu\nu}_{\ \ \alpha}\hat D^{\text{adj}}_\beta\hat F_{\mu\nu}\star\hat\psi,\\ \hat D^{\text{adj}}_\beta\hat F_{\mu\nu} &= \partial_\beta\hat F_{\mu\nu} - \mathrm{i}[\hat A_\beta,\hat F_{\mu\nu}]_\star,\end{aligned} \tag{4.50}$$

where $\hat\Sigma_{\text{cl}}$ was given in (2.3) and g_e is a new coupling constant. It turns out that all divergences generated by this extension term are – apart from the trivial one due to the wave function renormalisation of $\bar\psi,\psi$ – proportional to the electron mass m. In other words, in massless non-commutative QED the θ-expansion of (4.50) is one-loop renormalisable up to first order in θ by the standard QED wave function and electron charge renormalisations, the renormalisation

$$g_e(\varepsilon) = g_e + \frac{3}{4}\frac{\hbar g^2}{(4\pi)^2\varepsilon} \tag{4.51}$$

of the additional coupling constant g_e and field redefinitions.

5 General considerations: Change of variables

In this section we further analyse NCYM theory expanded in θ. In the following we shall leave the option open as to whether fermions are included or not. Our starting point is a trivial expansion of (2.3) according to

$$(f\star g)(x) = f(x)g(x)$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}}{2}\right)^{n}\theta^{\alpha_1\beta_1}\dots\theta^{\alpha_n\beta_n}(\partial_{\alpha_1}\dots\partial_{\alpha_n}f(x))\\
&\quad\times(\partial_{\beta_1}\dots\partial_{\beta_n}g(x)),\\
&\hat{\Phi}_i=\Phi'_i,\quad\forall\hat{\Phi}_i,
\end{aligned}
\tag{5.1}
$$

where $\hat{\Phi}_i$ denotes *all* fields of the theory, with the index i labelling spin and particle type. We reconsider the Seiberg–Witten map

$$\Phi'_i=\Phi_i+\Omega_i[\Phi],\tag{5.2}$$

where the field polynomial $\Omega_i[\Phi]$ is at least linear in θ, as a change of integration variables in the path-integral

$$Z[J]=\mathcal{N}\int[\mathcal{D}\Phi']\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{cl}}[\Phi']+\frac{\mathrm{i}}{\hbar}J^i\Phi'_i\right).\tag{5.3}$$

Here, $\mathcal{N}$ is an (ill-defined) normalisation factor and $\Gamma_{\mathrm{cl}}[\Phi']$ is the gauge-fixed NCYM action – possibly including fermions – expanded according to (5.1) in θ. To improve the readability we omit space-time integrals in $J^i\Phi'_i\equiv\int\mathrm{d}^4xJ^i(x)\Phi'_i(x)$ as well as in the sequel. We apply (5.2) to (5.3) and find

$$
\begin{aligned}
Z[J]&=\mathcal{N}\int[\mathcal{D}\Phi]\det\left[\frac{\delta\Phi'_j}{\delta\Phi_k}\right]\\
&\quad\times\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{cl}}[\Phi'[\Phi]]+\frac{\mathrm{i}}{\hbar}J^i\Phi'_i[\Phi]\right)\\
&=\mathcal{N}\int[\mathcal{D}\Phi][\mathcal{D}\mathcal{C}][\mathcal{D}\bar{\mathcal{C}}]\\
&\quad\times\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{cl}}[\Phi'[\Phi]]+\bar{\mathcal{C}}^i\frac{\delta\Phi'_i}{\delta\Phi_j}\mathcal{C}_j+\frac{\mathrm{i}}{\hbar}(J^i\Phi_i\right.\\
&\quad\left.\left.+J^i\Omega_i[\Phi]+\bar{\mathcal{C}}^i\mathcal{J}_i+\bar{\mathcal{J}}^i\mathcal{C}_i)\right)\right|_{\mathcal{J}=\bar{\mathcal{J}}=0}\\
&\equiv\mathcal{N}\int[\mathcal{D}\Phi][\mathcal{D}\mathcal{C}][\mathcal{D}\bar{\mathcal{C}}]\exp\left(\frac{\mathrm{i}}{\hbar}\tilde{\Gamma}_{\mathrm{cl}}[\Phi,\mathcal{C},\bar{\mathcal{C}}]+\frac{\mathrm{i}}{\hbar}(J^i\Phi_i\right.\\
&\quad\left.\left.+J^i\Omega_i[\Phi]+\bar{\mathcal{C}}^i\mathcal{J}_i+\bar{\mathcal{J}}^i\mathcal{C}_i)\right)\right|_{\mathcal{J}=\bar{\mathcal{J}}=0}.
\end{aligned}
\tag{5.4}
$$

The ghosts and anti-ghosts $\mathcal{C}_i$ and $\bar{\mathcal{C}}_i$ are to be understood as "towers" of fields of mixed Grassmann grading according to the actual field they couple to. The effect of the new ghost sector and of the additional $J\Omega$ vertex introduced in (5.4) is of course to compensate for the performed field redefinition in agreement with the equivalence theorem [29–31].

As usual we split $\tilde{\Gamma}_{\mathrm{cl}}[\Phi,\mathcal{C},\bar{\mathcal{C}}]=\tilde{\Gamma}_{\mathrm{bil}}[\Phi,\mathcal{C},\bar{\mathcal{C}}]+\tilde{\Gamma}_{\mathrm{int}}[\Phi,\mathcal{C},\bar{\mathcal{C}}]$ into the bilinear part

$$\tilde{\Gamma}_{\mathrm{bil}}[\Phi,\mathcal{C},\bar{\mathcal{C}}]=-\frac{1}{2}\Phi_i(\Delta^{-1})^{ij}\Phi_j+\frac{\hbar}{\mathrm{i}}\bar{\mathcal{C}}^i\mathcal{C}_i\tag{5.5}$$

and an interaction part $\tilde{\Gamma}_{\mathrm{int}}[\Phi,\mathcal{C},\bar{\mathcal{C}}]$, in which the fields are replaced by functional derivatives with respect to the sources. Then the functional integration can (formally) be performed and yields

$$
\begin{aligned}
Z[J]&=\mathcal{N}'\exp\left(\frac{\mathrm{i}}{\hbar}J^i\Omega_i\left[\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial J}\right]\right)\\
&\quad\times\exp\left(\frac{\mathrm{i}}{\hbar}\tilde{\Gamma}_{\mathrm{int}}\left[\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial J},\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\bar{\mathcal{J}}},\pm\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\mathcal{J}}\right]\right)\\
&\quad\times\exp\left.\left(\frac{\mathrm{i}}{2\hbar}J^i\Delta_{ij}J^j-\left(\frac{\mathrm{i}}{\hbar}\right)^2\bar{\mathcal{J}}^i\mathcal{J}_i\right)\right|_{\mathcal{J}=\bar{\mathcal{J}}=0}.
\end{aligned}
\tag{5.6}
$$

The source J^i in front of Ω_i is external and therefore must not be differentiated. We can write J^i however as $(\Delta^{-1})^{ij}\Phi_j$ with $\Phi_j=(\hbar/\mathrm{i})(\delta/\delta J^j)$ and correct the error due to contractions of J^j with other sources. One type of these contractions is given by a loop of these $J^i\Omega_i$ vertices in the form

$$\frac{\delta\Omega_{i_1}}{\delta\Phi_{i_2}}\frac{\delta\Omega_{i_2}}{\delta\Phi_{i_3}}\dots\frac{\delta\Omega_{i_{n-1}}}{\delta\Phi_{i_n}}\frac{\delta\Omega_{i_n}}{\delta\Phi_{i_1}}.\tag{5.7}$$

These loops cancel exactly the ghost loops, because the ghost vertices are given by $\bar{\mathcal{C}}^i(\delta\Omega_i/\delta\Phi_j)\mathcal{C}_j$ and the ghost propagator equals 1. Next a single $J^i\Omega_i$ vertex can be contracted with $\tilde{\Gamma}_{\mathrm{int}}$ to give $-\Omega_i(\tilde{\Gamma}_{\mathrm{int}}/\delta\Phi_i)$. This new vertex can further be contracted, as well as the $\Omega_i(\Delta^{-1})^{ij}\Phi_j$ vertex, and we finally get

$$
\begin{aligned}
Z[J]&=\mathcal{N}'\exp\frac{\mathrm{i}}{\hbar}(\Gamma_{\mathrm{bil}}[\Phi-\Omega[\Phi-\Omega[\Phi-\dots]]]-\Gamma_{\mathrm{bil}}[\Phi]\\
&\quad\left.+\tilde{\Gamma}_{\mathrm{int}}[\Phi-\Omega[\Phi-\Omega[\Phi-\dots]]])\right|_{\Phi\mapsto(\hbar/\mathrm{i})\delta/\delta J}\\
&\quad\times\exp\left(\frac{\mathrm{i}}{2\hbar}J^i\Delta_{ij}J^j\right).
\end{aligned}
\tag{5.8}
$$

Recalling $\Phi'=\Phi+\Omega[\Phi]$ and $\Gamma[\Phi+\Omega[\Phi]]=\tilde{\Gamma}[\Phi]$, (5.8) simplifies to the formula obtained by a direct computation of (5.3), i.e. without the change of variables (5.2),

$$Z[J]=\mathcal{N}'\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{int}}\left[\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial J}\right]\right)\exp\left(\frac{\mathrm{i}}{2\hbar}J^i\Delta_{ij}J^j\right).\tag{5.9}$$

The equivalence of (5.6) and (5.9) was of course expected. We are, however, interested in a different question. It is clear that (5.9) yields the (general) Green's functions of Case II, but how can we relate it to the Green's functions of [22] and Case I?

To answer this question we pass to the generating functional

$$Z_c[J]=\frac{\hbar}{\mathrm{i}}\ln Z[J]\tag{5.10}$$

of connected Green's functions and by Legendre transformation to the generating functional

$$\Gamma[\Phi_{\mathrm{cl}}]=Z_c[J]-J^i\Phi_{i,\mathrm{cl}}\tag{5.11}$$

of 1PI Green's functions, where J^i has to be replaced by the inverse solution of

$$\Phi_{i,\mathrm{cl}}=\frac{\delta Z_c[J]}{\delta J^i}.\tag{5.12}$$

In this way $\Gamma[\Phi_{\rm cl}]$ is obtained as a formal sum over ℓ-loop Feynman graphs. The model studied in [22] is given by the *subset* of Feynman graphs corresponding to (5.6) but *without closed* $(\mathcal{C},\bar{\mathcal{C}})$*-ghost loops and without the vertices involving* Ω. The Case I Feynman graphs are obtained by leaving out the fermionic part of the Ω vertex and the corresponding ghosts. We show now that 1PI Graphs in $Z[J]$ involving a single Ω vertex result in a field redefinition, but this property does not extend to higher order in Ω.

The 1PI-part of $Z_c[J]$ which is at most linear in Ω has the form

$$Z_c^{1\mathrm{PI,lin}(\Omega)}[J] = \frac{1}{2}J^i\Delta_{ij}J^j + \tilde{\Gamma}_{\rm int}[\Delta{\cdot}J] + \tilde{\Gamma}_{\rm eff}^{(\ell\geq 1)}[\Delta{\cdot}J] + J^i\Omega_{{\rm eff},i}^{(\ell\geq 0)}[\Delta{\cdot}J], \tag{5.13}$$

where $(\Delta{\cdot}J)_i = \Delta_{ij}J^j$. All $(\ell{\geq}1)$-loop 1PI graphs without the Ω vertex are contained in $\Gamma_{\rm eff}^{(\ell\geq 1)}$ and all 1PI graphs involving the Ω vertex are contained in $\Omega_{{\rm eff},i}^{(\ell\geq 0)}$. All graphs are built with the $\tilde{\Gamma}_{\rm int}$ vertices and $(\bar{\mathcal{C}},\mathcal{C})$-ghost loops are omitted, assuming the ghost tadpole $\delta\Omega_i/\delta\Phi_i$ in (5.7) to be zero. Now we obtain

$$\begin{aligned}
\Phi_{i,\rm cl} &= (\Delta{\cdot}J)_i + \Delta_{ij}\frac{\delta\tilde{\Gamma}_{\rm int}}{\delta\Phi_j}[\Delta{\cdot}J] \\
&\quad + \Delta_{ij}\frac{\delta\tilde{\Gamma}_{\rm eff}^{(\ell\geq 1)}}{\delta\Phi_j}[\Delta{\cdot}J] + \Delta_{ij}J^k\frac{\delta\Omega_{{\rm eff},k}^{(\ell\geq 0)}}{\delta\Phi_j}[\Delta{\cdot}J] \\
&\quad + \Omega_{{\rm eff},i}^{(\ell\geq 0)}[\Delta{\cdot}J], \qquad (5.14)\\
(\Delta{\cdot}J)_i^{\mathrm{lin}(\Omega)} &= \Phi_{i,\rm cl} - \Delta_{ij}\frac{\delta\tilde{\Gamma}_{\rm int}}{\delta\Phi_j}[\Phi_{\rm cl}] - \Delta_{ij}\frac{\delta\tilde{\Gamma}_{\rm eff}^{(\ell\geq 1)}}{\delta\Phi_j}[\Phi_{\rm cl}] \\
&\quad + \Delta_{ij}\frac{\delta^2\tilde{\Gamma}_{\rm int}}{\delta\Phi_j\delta\Phi_k}[\Phi_{\rm cl}]\Omega_{{\rm eff},k}^{(\ell\geq 0)}[\Phi_{\rm cl}] \\
&\quad + \Delta_{ij}\frac{\delta^2\tilde{\Gamma}_{\rm eff}^{(\ell\geq 1)}}{\delta\Phi_j\delta\Phi_k}[\Phi_{\rm cl}]\,\Omega_{{\rm eff},k}^{(\ell\geq 0)}[\Phi_{\rm cl}] - \Omega_{{\rm eff},i}^{(\ell\geq 0)}[\Phi_{\rm cl}] \\
&\quad - \Delta_{ij}J^k\frac{\delta\Omega_{{\rm eff},k}^{(\ell\geq 0)}}{\delta\Phi_j}[\Phi_{\rm cl}] + \text{1PR-terms}, \qquad (5.15)\\
\Gamma^{\mathrm{lin}(\Omega)}[\Phi_{\rm cl}] &= \Bigg(\tilde{\Gamma}_{\rm eff}^{(\ell\geq 0)}[\Delta{\cdot}J] \\
&\quad - (\Delta{\cdot}J)_i\frac{\delta\tilde{\Gamma}_{\rm int}}{\delta\Phi_i}[\Delta{\cdot}J] - (\Delta{\cdot}J)_i\frac{\delta\tilde{\Gamma}_{\rm eff}^{(\ell\geq 1)}}{\delta\Phi_i}[\Delta{\cdot}J] \\
&\quad - (\Delta{\cdot}J)_iJ^k\frac{\delta\Omega_{{\rm eff},k}^{(\ell\geq 0)}}{\delta\Phi_i}[\Delta{\cdot}J]\Bigg)^{1\mathrm{PI,lin}(\Omega)} \\
&= \tilde{\Gamma}_{\rm eff}^{(\ell\geq 0)}[\Phi_{\rm cl}] \\
&\quad - \frac{\delta\tilde{\Gamma}_{\rm eff}^{(\ell\geq 0)}}{\delta\Phi_i}[\Phi_{\rm cl}]\Omega_{{\rm eff},i}^{(\ell\geq 0)}[\Phi_{\rm cl}]. \qquad (5.16)
\end{aligned}$$

Terms like $(\Phi_{i,\rm cl} - \Omega_{{\rm eff},i}^{(\ell\geq 0)})(\delta\Gamma_{\rm int}/\delta\Phi_i)[\Phi_{\rm cl}]$ cancel via the direct occurrence in the first line of (5.16) and the substitution of (5.15) in $\Gamma_{\rm bil}[\Delta{\cdot}J] = -(1/2)(\Delta{\cdot}J)\Delta^{-1}(\Delta{\cdot}J)$. The final result (5.16) shows that graphs involving the Ω vertices in (5.6) linearly are a field redefinition. In our $(n=1,\ell=1)$-loop calculation the Ω vertices contribute already with $\ell=1$, therefore, the effect at total loop order 1 is expected to be

$$\Omega_{{\rm eff},i}^{(\ell=1)}\frac{\delta\Gamma_{\rm cl}}{\delta\Phi_i}, \tag{5.17}$$

which is exactly the difference of the loop calculations of [22], Case I and Case II.

Taking graphs with more than one Ω vertex into account, the difference of the cases under consideration cannot be a field redefinition any longer. Namely, there is now a graph $J^{i_1}\cdots J^{i_n}\Omega_{{\rm eff},i_1\dots i_n}^{(\ell\geq 1)}[\Delta{\cdot}J]$ in the generalisation of (5.13), which gives the term $(1{-}n)J^{i_1}\cdots J^{i_n}\Omega_{{\rm eff},i_1\dots i_n}^{(\ell\geq 1)}[\Delta{\cdot}J]$ in Γ. The free sources J^{i_k} are now replaced e.g. by $\delta\Gamma_{\rm eff}/\delta\Phi_k$ and thus lead to 1PI graphs where the Ω vertices become *inner*. These graphs cannot be reached by field redefinitions, which are *outer*. In conclusion, we expect at order θ^2 that the differences between [22], Case I and Case II are no longer field redefinitions.

In principle there are also the $(\mathcal{C},\bar{\mathcal{C}})$-ghost loops to take into account. However, the corresponding ghost propagator equals 1 and the ghost couplings are polynomial in momenta and masses. If there are no sub-divergences, all ghost loops vanish trivially, at least in analytic and dimensional regularisation. Accordingly, if the $(\mathcal{C},\bar{\mathcal{C}})$-ghost vertices are renormalisable, the $(\mathcal{C},\bar{\mathcal{C}})$-ghosts give no contribution at all.

6 Discussion

In this paper we have continued the quantum analysis of the Seiberg–Witten map first carried out in [20–22]. We have analysed θ-expanded non-commutative QED, which happens to be the easiest non-commutative model to study in this context. In contrast to [22], where both bosonic and fermionic sectors were θ-expanded via the Seiberg–Witten differential equations, we have analysed in this paper the two cases where

(I) only the bosonic sector is expanded via the Seiberg–Witten map, and
(II) neither the bosonic nor the fermionic sectors are expanded via the Seiberg–Witten map.

We have found that up to field redefinitions the outcome of all three approaches is identical. We can summarise our picture about the Seiberg–Witten map as follows:

(1) The Seiberg–Witten expansion must be seen as a true (physical) expansion of the fields in a gauge theory, which is performed *prior* to quantisation. Otherwise (expanding after the quantisation) ghosts and Ω vertices generated due to the change of integration variables would contribute to the loop calculation and lead to the same result as without the Seiberg–Witten map.
(2) At first order in θ no difference between θ-expanded quantum field theories with and without Seiberg–Witten

map is expected (apart from problems with the choice of the gauge group, which we ignore here). Our one-loop QED calculations confirm this.
(3) θ-expanded gauge theory can*not* be expected to be stable under quantisation because divergences will appear already at first order in θ, for the reason that no symmetry is known which rules them out. At first order in θ the additional terms added to the initial action in order to have enough freedom to absorb these divergences are the same when using the Seiberg–Witten map or leaving it out.
(4) At second order in θ there will be substantial differences between θ-expansion with or without Seiberg–Witten map due to contributions of the Ω vertices (and possibly non-renormalisable ghost sub-divergences).

The most important result of this paper is perhaps that *if* one insists on analysing θ-expanded (Abelian) gauge theories involving fermions one *must* add the term

$$g_e \int \mathrm{d}^4x \mathrm{i}\theta^{\alpha\beta}\hat{\bar{\psi}} \star \gamma^{\mu\nu}_{\alpha} \hat{D}^{\mathrm{adj}}_{\beta} \hat{F}_{\mu\nu} \star \hat{\psi}$$

to the non-expanded initial action. Also, the fermion masses should be introduced via a Higgs mechanism.

Let us finally stress that it is not yet possible to make definite conclusions towards renormalisability of θ-expanded models. It appears that explicit loop calculations at second order in θ are needed, these are however not easily accessible due to the enormous volume of calculations involved.

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